

# THE GEOMETRY OF LOOP SPACES I: $H^s$ -RIEMANNIAN METRICS

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**ABSTRACT.** A Riemannian metric on a manifold  $M$  induces a family of Riemannian metrics on the loop space  $LM$  depending on a Sobolev space parameter  $s$ . We compute the connection forms of these metrics and the higher symbols of their curvature forms, which take values in pseudodifferential operators ( $\Psi$ DOs). These calculations are used in the followup paper [10] to construct Chern-Simons classes on  $TLM$  which detect nontrivial elements in the diffeomorphism group of certain Sasakian 5-manifolds associated to Kähler surfaces.

Dedicated to the memory of Prof. Shoshichi Kobayashi

## 1. Introduction

The loop space  $LM$  of a manifold  $M$  appears frequently in mathematics and mathematical physics. In this paper, we develop the Riemannian geometry of loop spaces. In a companion paper [10], we describe a computable theory of characteristic classes for the tangent bundle  $TLM$ .

Several new features appear for Riemannian geometry on the infinite dimensional manifold  $LM$ . First, for a fixed Riemannian metric  $g$  on  $M$ , there is a natural one-parameter family of metrics  $g^s$  on  $LM$  associated to a Sobolev parameter  $s > 0$ . Our main goal is to compute the Levi-Civita connection for  $g^s$  and the associated connection and curvature forms. For  $s = 0$ , this is the usual  $L^2$  metric, whose connection one-form and curvature two-form are essentially the same as the corresponding forms for  $g$ . Second, for  $s > 0$  these forms take values in zeroth order pseudodifferential operators ( $\Psi$ DOs) acting on sections of a trivial bundle over the circle. Thus the analysis of these  $\Psi$ DOs and their symbols is essential to understanding the geometry of  $LM$ . In contrast to the finite dimensional case, where a Riemannian metric on  $M$  implements a *reduction* of the structure group of  $TM$ ,  $g^s$  is not compatible with the structure group of  $TLM$ . This forces an *extension* of the structure group of  $TLM$  from a gauge group to a group of invertible zeroth order  $\Psi$ DOs.

The paper is organized as follows. §2 discusses connections associated to  $g^s$ . After some preliminary material on  $\Psi$ DOs, we compute the Levi-Civita connection for

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$s = 0$  (Lemma 2.1),  $s = 1$  (Theorem 2.2),  $s \in \mathbb{Z}^+$  (Theorem 2.10), and general  $s > \frac{1}{2}$  (Theorem 2.12). The extension of the structure group is discussed in §2.6. In §3, we show that our results extend work of Freed and Larrain-Hubach on loop groups [3, 8]. In the Appendix, we compute the higher order symbols of the connection and curvature forms of these connections.

One main motivation for this paper is the construction of characteristic and secondary classes on  $TLM$ , using the Wodzicki residue of  $\Psi$ DOs. In the companion paper [10], we use the main theorems in this paper to construct Chern-Simons classes on  $TLM$  which detect that  $\pi_1(\text{Diff}(M^5))$  is infinite, where  $\text{Diff}(M^5)$  is the diffeomorphism group of infinite families of Sasakian 5-manifolds associated to integral Kähler surfaces. Thus this work relates to a major theme in Prof. Kobayashi's many important papers, namely the relationship between Riemannian and complex geometry.

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## 2. The Levi-Civita Connection for Sobolev Parameter $s \geq 0$

In this section, we compute the Levi-Civita connection on  $LM$  associated to a Riemannian metric on  $M$  and a Sobolev parameter  $s = 0$  or  $s > \frac{1}{2}$ . The standard  $L^2$  metric on  $LM$  is the case  $s = 0$ , and otherwise we avoid technical issues by assuming that  $s$  is greater than the critical exponent  $\frac{1}{2}$  for analysis on bundles over  $S^1$ . The main results are Lemma 2.1, Theorem 2.2, Theorem 2.10, and Theorem 2.12, which compute the Levi-Civita connection for  $s = 0$ ,  $s = 1$ ,  $s \in \mathbb{Z}^+$ , and general  $s > \frac{1}{2}$ , respectively.

### 2.1. Preliminaries on $LM$ .

Let  $(M, \langle \cdot, \cdot \rangle)$  be a closed, connected, oriented Riemannian  $n$ -manifold with loop space  $LM = C^\infty(S^1, M)$  of smooth loops.  $LM$  is a smooth infinite dimensional Fréchet manifold, but it is technically simpler to work with the smooth Hilbert manifold  $H^{s'}(S^1, M)$  of loops in some Sobolev class  $s' \gg 0$ , as we now recall. For  $\gamma \in LM$ , the formal tangent space  $T_\gamma LM$  is  $\Gamma(\gamma^*TM)$ , the space of smooth sections of the pullback bundle  $\gamma^*TM \rightarrow S^1$ . The actual tangent space of  $H^{s'}(S^1, M)$  at  $\gamma$  is  $H^{s'-1}(\gamma^*TM)$ , the sections of  $\gamma^*TM$  of Sobolev class  $s' - 1$ . We will fix  $s'$  and use  $LM, T_\gamma LM$  for  $H^{s'}(S^1, M), H^{s'-1}(\gamma^*TM)$ , respectively.

For each  $s > 1/2$ , we can complete  $\Gamma(\gamma^*TM \otimes \mathbb{C})$  with respect to the Sobolev inner product

$$\langle X, Y \rangle_s = \frac{1}{2\pi} \int_0^{2\pi} \langle (1 + \Delta)^s X(\theta), Y(\theta) \rangle_{\gamma(\theta)} d\theta, \quad X, Y \in \Gamma(\gamma^*TM). \quad (2.1)$$

Here  $\Delta = D^*D$ , with  $D = D/d\gamma$  the covariant derivative along  $\gamma$ . (We use this notation instead of the classical  $D/d\theta$  to keep track of  $\gamma$ .) We need the complexified pullback bundle  $\gamma^*TM \otimes \mathbb{C}$ , denoted from now on just as  $\gamma^*TM$ , in order to apply the pseudodifferential operator  $(1 + \Delta)^s$ . The construction of  $(1 + \Delta)^s$  is reviewed in

§2.2. By the basic elliptic estimate, the completion of  $\gamma^*TM$  with respect to (2.1) is  $H^s(\gamma^*TM)$ . We can consider the  $s$  metric on  $TLM$  for any  $s \in \mathbb{R}$ , but we will only consider  $s = 0$  or  $1/2 < s \leq s' - 1$ .

A small real neighborhood  $U_\gamma$  of the zero section in  $H^{s'-1}(\gamma^*TM)$  is a coordinate chart near  $\gamma \in LM$  via the pointwise exponential map

$$\exp_\gamma : U_\gamma \longrightarrow LM, \quad X \mapsto (\theta \mapsto \exp_{\gamma(\theta)} X(\theta)). \quad (2.2)$$

The differentiability of the transition functions  $\exp_{\gamma_1}^{-1} \cdot \exp_{\gamma_2}$  is proved in [2] and [4, Appendix A]. Here  $\gamma_1, \gamma_2$  are close loops in the sense that a geodesically convex neighborhood of  $\gamma_1(\theta)$  contains  $\gamma_2(\theta)$  and vice versa for all  $\theta$ . Since  $\gamma^*TM$  is (noncanonically) isomorphic to the trivial bundle  $\mathcal{R} = S^1 \times \mathbb{C}^n \longrightarrow S^1$ , the model space for  $LM$  is the set of  $H^{s'}$  sections of this trivial bundle. The  $s$  metric is a weak Riemannian metric for  $s < s' - 1$  in the sense that the topology induced on  $H^{s'}(S^1, M)$  by the exponential map applied to  $H^s(\gamma^*TM)$  is weaker than the  $H^{s'}$  topology.

The complexified tangent bundle  $TLM$  has transition functions  $d(\exp_{\gamma_1}^{-1} \circ \exp_{\gamma_2})$ . Under the isomorphisms  $\gamma_1^*TM \simeq \mathcal{R} \simeq \gamma_2^*TM$ , the transition functions lie in the gauge group  $\mathcal{G}(\mathcal{R})$ , so this is the structure group of  $TLM$ .

## 2.2. Review of $\Psi$ DO Calculus.

We recall the construction of classical pseudodifferential operators ( $\Psi$ DOs) on a closed manifold  $M$  from [5, 13], assuming knowledge of  $\Psi$ DOs on  $\mathbb{R}^n$  (see e.g. [6, 14]).

A linear operator  $P : C^\infty(M) \longrightarrow C^\infty(M)$  is a  $\Psi$ DO of order  $d$  if for every open chart  $U \subset M$  and functions  $\phi, \psi \in C_c^\infty(U)$ ,  $\phi P \psi$  is a  $\Psi$ DO of order  $d$  on  $\mathbb{R}^n$ , where we do not distinguish between  $U$  and its diffeomorphic image in  $\mathbb{R}^n$ . Let  $\{U_i\}$  be a finite cover of  $M$  with subordinate partition of unity  $\{\phi_i\}$ . Let  $\psi_i \in C_c^\infty(U_i)$  have  $\psi_i \equiv 1$  on  $\text{supp}(\phi_i)$  and set  $P_i = \psi_i P \phi_i$ . Then  $\sum_i \phi_i P_i \psi_i$  is a  $\Psi$ DO on  $M$ , and  $P$  differs from  $\sum_i \phi_i P_i \psi_i$  by a smoothing operator, denoted  $P \sim \sum_i \phi_i P_i \psi_i$ . In particular, this sum is independent of the choices up to smoothing operators. All this carries over to  $\Psi$ DOs acting on sections of a bundle over  $M$ .

An example is the  $\Psi$ DO  $(1 + \Delta - \lambda)^{-1}$  for  $\Delta$  a positive order nonnegative elliptic  $\Psi$ DO and  $\lambda$  outside the spectrum of  $1 + \Delta$ . In each  $U_i$ , we construct a parametrix  $P_i$  for  $A_i = \psi_i(1 + \Delta - \lambda)\phi_i$  by formally inverting  $\sigma(A_i)$  and then constructing a  $\Psi$ DO with the inverted symbol. By [1, App. A],  $B = \sum_i \phi_i P_i \psi_i$  is a parametrix for  $(1 + \Delta - \lambda)^{-1}$ . Since  $B \sim (1 + \Delta - \lambda)^{-1}$ ,  $(1 + \Delta - \lambda)^{-1}$  is itself a  $\Psi$ DO. For  $x \in U_i$ , by definition

$$\sigma((1 + \Delta - \lambda)^{-1})(x, \xi) = \sigma(P)(x, \xi) = \sigma(\phi P \phi)(x, \xi),$$

where  $\phi$  is a bump function with  $\phi(x) = 1$  [5, p. 29]; the symbol depends on the choice of  $(U_i, \phi_i)$ .

The operator  $(1 + \Delta)^s$  for  $\text{Re}(s) < 0$ , which exists as a bounded operator on  $L^2(M)$  by the functional calculus, is also a  $\Psi$ DO. To see this, we construct the putative symbol  $\sigma_i$  of  $\psi_i(1 + \Delta)^s\phi_i$  in each  $U_i$  by a contour integral  $\int_\Gamma \lambda^s \sigma[(1 + \Delta - \lambda)^{-1}] d\lambda$

around the spectrum of  $1 + \Delta$ . We then construct a  $\Psi$ DO  $Q_i$  on  $U_i$  with  $\sigma(Q_i) = \sigma_i$ , and set  $Q = \sum_i \phi_i Q_i \psi_i$ . By arguments in [13],  $(1 + \Delta)^s \sim Q$ , so  $(1 + \Delta)^s$  is a  $\Psi$ DO.

### 2.3. The Levi-Civita Connection for $s = 0, 1$ .

The smooth Riemannian manifold  $LM = H^{s'}(S^1, M)$  has tangent bundle  $TLM$  with  $T_\gamma LM = H^{s'-1}(\gamma^* TM)$ . For the  $s' - 1$  metric on  $TLM$  (i.e.,  $s = s' - 1$  in (2.1)), the Levi-Civita connection exists and is determined by the six term formula

$$\begin{aligned} 2\langle \nabla_X^s Y, Z \rangle_s &= X\langle Y, Z \rangle_s + Y\langle X, Z \rangle_s - Z\langle X, Y \rangle_s \\ &\quad + \langle [X, Y], Z \rangle_s + \langle [Z, X], Y \rangle_s - \langle [Y, Z], X \rangle_s \end{aligned} \quad (2.3)$$

[7, Ch. VIII]. The point is that each term on the RHS of (2.3) is a *continuous* linear functional  $T_i : H^{s=s'-1}(\gamma^* TM) \rightarrow \mathbb{C}$  in  $Z$ . Thus  $T_i(Z) = \langle T'_i(X, Y), Z \rangle_s$  for a unique  $T'(X, Y) \in H^{s'-1}(\gamma^* TM)$ , and  $\nabla_Y^s X = \frac{1}{2} \sum_i T'_i$ .

In general, the Sobolev parameter  $s$  in (2.1) differs from the parameter  $s'$  defining the loop space. We discuss how this affects the existence of a Levi-Civita connection.

**Remark 2.1.** For general  $s > \frac{1}{2}$ , the Levi-Civita connection for the  $H^s$  metric is guaranteed to exist on the bundle  $H^s(\gamma^* TM)$ , as above. However, it is inconvenient to have the bundle depend on the Sobolev parameter, for several reasons: (i)  $H^s(\gamma^* TM)$  is strictly speaking not the tangent bundle of  $LM$ , (ii) for the  $L^2$  ( $s = 0$ ) metric, the Levi-Civita connection should be given by the Levi-Civita connection on  $M$  applied pointwise along the loop (see Lemma 2.1), and on  $L^2(\gamma^* TM)$  this would have to be interpreted in the distributional sense; (iii) to compute Chern-Simons classes on  $LM$  in [10], we need to compute with a pair of connections corresponding to  $s = 0, s = 1$  on the same bundle. These problems are not fatal: (i) and (ii) are essentially aesthetic issues, and for (iii), the connection one-forms will take values in zeroth order  $\Psi$ DOs, which are bounded operators on any  $H^{s'-1}(\gamma^* TM)$ , so  $s' \gg 0$  can be fixed.

Thus it is more convenient to fix  $s'$  and consider the family of  $H^s$  metrics on  $TLM$  for  $\frac{1}{2} < s < s' - 1$ . However, the existence of the Levi-Civita connection for the  $H^s$  metric is trickier. For a sequence  $Z \in H^{s'-1} = H^{s'-1}(\gamma^* TM)$  with  $Z \rightarrow 0$  in  $H^{s'-1}$  or in  $H^s$ , the RHS of (2.3) goes to 0 for fixed  $X, Y \in H^s$ . Since  $H^{s'-1}$  is dense in  $H^s$ , the RHS of (2.3) extends to a continuous linear functional on  $H^s$ . Thus the RHS of (2.3) is given by  $\langle L(X, Y), Z \rangle_s$  for some  $L(X, Y) \in H^s$ . We set  $\nabla_Y^s X = \frac{1}{2} L(X, Y)$ . Note that even if we naturally demand that  $X, Y \in H^{s'-1}$ , we only get  $\nabla_Y^s X \in H^s \supset H^{s'-1}$  without additional work. Part of the content of Theorem 2.12 is that the Levi-Civita connection exists in the *strong sense*: given a tangent vector  $X \in H^{s'-1}(\gamma^* TM)$  and a smooth vector field  $Y_\eta \in H^{s'-1}(\eta^* TM)$  for all  $\eta$ ,  $\nabla_X^s Y(\gamma) \in H^{s'-1}(\gamma^* TM)$ . See Remark 2.6.

We need to discuss local coordinates on  $LM$ . For motivation, recall that

$$[X, Y]^a = X(Y^a) \partial_a - Y(X^a) \partial_a \equiv \delta_X(Y) - \delta_Y(X) \quad (2.4)$$

in local coordinates on a finite dimensional manifold. Note that  $X^i \partial_i Y^a = X(Y^a) = (\delta_X Y)^a$  in this notation.

Let  $Y$  be a vector field on  $LM$ , and let  $X$  be a tangent vector at  $\gamma \in LM$ . The local variation  $\delta_X Y$  of  $Y$  in the direction of  $X$  at  $\gamma$  is defined as usual: let  $\gamma(\varepsilon, \theta)$  be a family of loops in  $M$  with  $\gamma(0, \theta) = \gamma(\theta)$ ,  $\frac{d}{d\varepsilon}|_{\varepsilon=0} \gamma(\varepsilon, \theta) = X(\theta)$ . Fix  $\theta$ , and let  $(x^a)$  be coordinates near  $\gamma(\theta)$ . We call these coordinates *manifold coordinates*. Then

$$\delta_X Y^a(\gamma)(\theta) \stackrel{\text{def}}{=} \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} Y^a(\gamma(\varepsilon, \theta)).$$

Note that  $\delta_X Y^a = (\delta_X Y)^a$  by definition.

**Remark 2.2.** Having  $(x^a)$  defined only near a fixed  $\gamma(\theta)$  is inconvenient. We can find coordinates that work for all points of  $\gamma(\theta)$  as follows. For fixed  $\gamma$ , there is an  $\varepsilon$  such that for all  $\theta$ ,  $\exp_{\gamma(\theta)} X$  is inside the cut locus of  $\gamma(\theta)$  if  $X \in T_{\gamma(\theta)} M$  has  $|X| < \varepsilon$ . Fix such an  $\varepsilon$ . Call  $X \in H^{s'-1}(\gamma^* TM)$  *short* if  $|X(\theta)| < \varepsilon$  for all  $\theta$ . Then

$$U_\gamma = \{\theta \mapsto \exp_{\gamma(\theta)} X(\theta) \mid X \text{ is short}\} \subset LM$$

is a coordinate neighborhood of  $\gamma$  parametrized by  $\{X : X \text{ is short}\}$ .

We know  $H^{s'-1}(\gamma^* TM) \simeq H^{s'-1}(S^1 \times \mathbb{R}^n)$  noncanonically, so  $U_\gamma$  is parametrized by short sections of  $H^{s'-1}(S^1 \times \mathbb{R}^n)$  for a different  $\varepsilon$ . In particular, we have a smooth diffeomorphism  $\beta$  from  $U_\gamma$  to short sections of  $H^{s'-1}(S^1 \times \mathbb{R}^n)$ .

Put coordinates  $(x^a)$  on  $\mathbb{R}^n$ , which we identify canonically with the fiber  $\mathbb{R}_\theta^n$  over  $\theta$  in  $S^1 \times \mathbb{R}^n$ . For  $\eta \in U_\gamma$ , we have  $\beta(\eta) = (\beta(\eta)^1(\theta), \dots, \beta(\eta)^n(\theta))$ . As with finite dimensional coordinate systems, we will drop  $\beta$  and just write  $\eta = (\eta(\theta)^a)$ . These coordinates work for all  $\eta$  near  $\gamma$  and for all  $\theta$ . The definition of  $\delta_X Y$  above carries over to exponential coordinates.

We will call these coordinates *exponential coordinates*.

(2.4) continues to hold for vector fields on  $LM$ , in either manifold or exponential coordinates. To see this, one checks that the coordinate-free proof that  $L_X Y(f) = [X, Y](f)$  for  $f \in C^\infty(M)$  (e.g. [16, p. 70]) carries over to functions on  $LM$ . In brief, the usual proof involves a map  $H(s, t)$  of a neighborhood of the origin in  $\mathbb{R}^2$  into  $M$ , where  $s, t$  are parameters for the flows of  $X, Y$ , resp. For  $LM$ , we have a map  $H(s, t, \theta)$ , where  $\theta$  is the loop parameter. The usual proof uses only  $s, t$  differentiations, so  $\theta$  is unaffected. The point is that the  $Y^i$  are local functions on the  $(s, t, \theta)$  parameter space, whereas the  $X^i$  are not local functions on  $M$  at points where loops cross or self-intersect.

We first compute the  $L^2$  ( $s = 0$ ) Levi-Civita connection invariantly and in manifold coordinates.

**Lemma 2.1.** *Let  $\nabla^{LC}$  be the Levi-Civita connection on  $M$ . Let  $\text{ev}_\theta : LM \rightarrow M$  be  $\text{ev}_\theta(\gamma) = \gamma(\theta)$ . Then  $D_X Y(\gamma)(\theta) \stackrel{\text{def}}{=} (\text{ev}_\theta^* \nabla^{LC})_X Y(\gamma)(\theta)$  is the  $L^2$  Levi-Civita*

connection on  $LM$ . In manifold coordinates,

$$(D_X Y)^a(\gamma)(\theta) = \delta_X Y^a(\gamma)(\theta) + \Gamma_{bc}^a(\gamma(\theta)) X^b(\gamma)(\theta) Y^c(\gamma)(\theta). \quad (2.5)$$

As in Remark 2.1, we may assume that  $X, Y \in H^{s'-1}(\gamma^* TM)$  with  $s' \gg 0$ , so (2.5) makes sense.

*Proof.*  $\text{ev}_\theta^* \nabla^{LC}$  is a connection on  $\text{ev}_\theta^* TM \rightarrow LM$ . We have  $\text{ev}_{\theta,*}(X) = X(\theta)$ . If  $U$  is a coordinate neighborhood on  $M$  near some  $\gamma(\theta)$ , then on  $\text{ev}_\theta^{-1}(U)$ ,

$$\begin{aligned} (\text{ev}_\theta^* \nabla^{LC})_X Y^a(\gamma)(\theta) &= (\delta_X Y)^a(\gamma)(\theta) + ((\text{ev}_\theta^* \omega_X^{LC}) Y)^a(\theta) \\ &= (\delta_X Y)^a(\gamma)(\theta) + \Gamma_{bc}^a(\gamma(\theta)) X^b(\gamma)(\theta) Y^c(\gamma)(\theta). \end{aligned}$$

Since  $\text{ev}_\theta^* \nabla^{LC}$  is a connection, for each fixed  $\theta$ ,  $\gamma$  and  $X \in T_\gamma LM$ ,  $Y \mapsto (\text{ev}_\theta^* \nabla^{LC})_X Y(\gamma)$  has Leibniz rule with respect to functions on  $LM$ . Thus  $D$  is a connection on  $LM$ .

$D$  is torsion free, as from the local expression  $D_X Y - D_Y X = \delta_X Y - \delta_Y X = [X, Y]$ .

To show that  $D_X Y$  is compatible with the  $L^2$  metric, first recall that for a function  $f$  on  $LM$ ,  $D_X f = \delta_X f = \frac{d}{d\varepsilon}|_{\varepsilon=0} f(\gamma(\varepsilon, \theta))$  for  $X(\theta) = \frac{d}{d\varepsilon}|_{\varepsilon=0} \gamma(\varepsilon, \theta)$ . (Here  $f$  depends only on  $\gamma$ .) Thus (suppressing the partition of unity, which is independent of  $\varepsilon$ )

$$\begin{aligned} D_X \langle Y, Z \rangle_0 &= \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \int_{S^1} g_{ab}(\gamma(\varepsilon, \theta)) Y^a(\gamma(\varepsilon, \theta)) Z^b(\gamma(\varepsilon, \theta)) d\theta \\ &= \int_{S^1} \partial_c g_{ab}(\gamma(\varepsilon, \theta)) X^c Y^a(\gamma(\varepsilon, \theta)) Z^b(\gamma(\varepsilon, \theta)) d\theta \\ &\quad + \int_{S^1} g_{ab}(\gamma(\varepsilon, \theta)) (\delta_X Y)^a(\gamma(\varepsilon, \theta)) Z^b(\gamma(\varepsilon, \theta)) d\theta \\ &\quad + \int_{S^1} g_{ab}(\gamma(\varepsilon, \theta)) Y^a(\gamma(\varepsilon, \theta)) (\delta_X Z)^b(\gamma(\varepsilon, \theta)) d\theta \\ &= \int_{S^1} \Gamma_{ca}^e g_{eb} X^c Y^a Z^b + \Gamma_{cb}^e g_{ae} X^c Y^a Z^b \\ &\quad + g_{ab} (\delta_X Y)^a Z^b + g_{ab} Y^a (\delta_X Z)^b d\theta \\ &= \langle D_X Y, Z \rangle_0 + \langle Y, D_X Z \rangle_0. \end{aligned}$$

□

**Remark 2.3.** The local expression for  $D_X Y$  also holds in exponential coordinates. More precisely, let  $(e_1(\theta), \dots, e_n(\theta))$  be a global frame of  $\gamma^* TM$  given by the trivialization of  $\gamma^* TM$ . Then  $(e_i(\theta))$  is also naturally a frame of  $T_X T_{\gamma(\theta)} M$  for all  $X \in T_{\gamma(\theta)} M$ . We use  $\exp_{\gamma(\theta)}$  to pull back the metric on  $M$  to a metric on  $T_{\gamma(\theta)} M$ :

$$g_{ij}(X) = (\exp_{\gamma(\theta)}^* g)(e_i, e_j) = g(d(\exp_{\gamma(\theta)})_X(e_i), d(\exp_{\gamma(\theta)})_X(e_j))_{\exp_{\gamma(\theta)} X}.$$

Then the Christoffel symbols  $\Gamma_{bc}^a(\gamma(\theta))$  the term *e.g.*, are computed with respect to this metric. For example, the term  $\partial_\ell g_{bc}$  means  $e_\ell g(e_a, e_b)$ , etc. The proof that  $D_X Y$  has the local expression (2.5) then carries over to exponential coordinates.

The  $s = 1$  Levi-Civita connection on  $LM$  is given as follows.

**Theorem 2.2.** *The  $s = 1$  Levi-Civita connection  $\nabla^1$  on  $LM$  is given at the loop  $\gamma$  by*

$$\begin{aligned} \nabla_X^1 Y &= D_X Y + \frac{1}{2}(1 + \Delta)^{-1} [-\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}}Y \\ &\quad - \nabla_{\dot{\gamma}}(R(Y, \dot{\gamma})X) - R(Y, \dot{\gamma})\nabla_{\dot{\gamma}}X \\ &\quad + R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} - R(\nabla_{\dot{\gamma}}X, Y)\dot{\gamma}]. \end{aligned}$$

On the right hand side of this formula, the term  $\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y)$  denotes the vector field along  $\gamma$  whose value at  $\theta$  is  $\nabla_{\dot{\gamma}(\theta)}^{LC} R(X(\theta), \dot{\gamma}(\theta))Y(\theta)$ .

We prove this in a series of steps. The assumption in the next Proposition will be dropped later.

**Proposition 2.3.** *The Levi-Civita connection for the  $s = 1$  metric is given by*

$$\nabla_X^1 Y = D_X Y + \frac{1}{2}(1 + \Delta)^{-1}[D_X, 1 + \Delta]Y + \frac{1}{2}(1 + \Delta)^{-1}[D_Y, 1 + \Delta]X + A_X Y,$$

where we assume that for  $X, Y \in H^{s'-1}$ ,  $A_X Y$  is well-defined by

$$-\frac{1}{2}\langle [D_Z, 1 + \Delta]X, Y \rangle_0 = \langle A_X Y, Z \rangle_1. \quad (2.6)$$

*Proof.* By Lemma 2.1,

$$\begin{aligned} X\langle Y, Z \rangle_1 &= X\langle (1 + \Delta)Y, Z \rangle_0 = \langle D_X((1 + \Delta)Y), Z \rangle_0 + \langle (1 + \Delta)Y, D_X Z \rangle_0 \\ Y\langle X, Z \rangle_1 &= \langle D_Y((1 + \Delta)X), Z \rangle_0 + \langle (1 + \Delta)X, D_Y Z \rangle_0 \\ -Z\langle X, Y \rangle_1 &= -\langle D_Z((1 + \Delta)X), Y \rangle_0 - \langle (1 + \Delta)X, D_Z Y \rangle_0 \\ \langle [X, Y], Z \rangle_1 &= \langle (1 + \Delta)(\delta_X Y - \delta_Y X), Z \rangle_0 = \langle (1 + \Delta)(D_X Y - D_Y X), Z \rangle_0 \\ \langle [Z, X], Y \rangle_1 &= \langle (1 + \Delta)(D_Z X - D_X Z), Y \rangle_0 \\ -\langle [Y, Z], X \rangle_1 &= -\langle (1 + \Delta)(D_Y Z - D_Z Y), X \rangle_0. \end{aligned}$$

The six terms on the left hand side must sum up to  $2\langle \nabla_X^1 Y, Z \rangle_1$  in the sense of Remark 2.1. After some cancellations, we get

$$\begin{aligned}
2\langle \nabla_X^1 Y, Z \rangle_1 &= \langle D_X((1 + \Delta)Y), Z \rangle_0 + \langle D_Y((1 + \Delta)X), Z \rangle_0 \\
&\quad + \langle (1 + \Delta)(D_X Y - D_Y X), Z \rangle_0 - \langle D_Z((1 + \Delta)X), Y \rangle_0 \\
&\quad + \langle (1 + \Delta)D_Z X, Y \rangle_0 \\
&= \langle (1 + \Delta)D_X Y, Z \rangle_0 + \langle [D_X, 1 + \Delta]Y, Z \rangle_0 \\
&\quad + \langle (1 + \Delta)D_Y X, Z \rangle_0 + \langle [D_Y, 1 + \Delta]X, Z \rangle_0 \\
&\quad + \langle (1 + \Delta)(D_X Y - D_Y X), Z \rangle_0 - \langle [D_Z, 1 + \Delta]X, Y \rangle_0 \\
&= 2\langle D_X Y, Z \rangle_1 + \langle (1 + \Delta)^{-1}[D_X, 1 + \Delta]Y, Z \rangle_1 \\
&\quad + \langle (1 + \Delta)^{-1}[D_Y, 1 + \Delta]X, Z \rangle_1 + 2\langle A_X Y, Z \rangle_1.
\end{aligned}$$

□

Now we compute the bracket terms in the Proposition. We have  $[D_X, 1 + \Delta] = [D_X, \Delta]$ . For  $\dot{\gamma} = (d/d\theta)\gamma$ ,

$$0 = \dot{\gamma}\langle X, Y \rangle_0 = \langle \nabla_{\dot{\gamma}} X, Y \rangle_0 + \langle X, \nabla_{\dot{\gamma}} Y \rangle_0,$$

so

$$\Delta = \nabla_{\dot{\gamma}}^* \nabla_{\dot{\gamma}} = -\nabla_{\dot{\gamma}}^2. \quad (2.7)$$

**Lemma 2.4.**  $[D_X, \nabla_{\dot{\gamma}}]Y = R(X, \dot{\gamma})Y$ .

*Proof.* Note that  $\gamma^\nu, \dot{\gamma}^\nu$  are locally defined functions on  $S^1 \times LM$ . Let  $\tilde{\gamma} : [0, 2\pi] \times (-\varepsilon, \varepsilon) \rightarrow M$  be a smooth map with  $\tilde{\gamma}(\theta, 0) = \gamma(\theta)$ , and  $\frac{d}{d\tau}|_{\tau=0}\tilde{\gamma}(\theta, \tau) = Z(\theta)$ . Since  $(\theta, \tau)$  are coordinate functions on  $S^1 \times (-\varepsilon, \varepsilon)$ , we have

$$\begin{aligned}
Z(\dot{\gamma}^\nu) &= \delta_Z(\dot{\gamma}^\nu) = \partial_\tau^Z(\dot{\gamma}^\nu) = \frac{\partial}{\partial \tau} \Big|_{\tau=0} \left( \frac{\partial}{\partial \theta}(\tilde{\gamma}(\theta, \tau)^\nu) \right) \\
&= \frac{\partial}{\partial \theta} \frac{\partial}{\partial \tau} \Big|_{\tau=0} \tilde{\gamma}(\theta, \tau)^\nu = \partial_\theta Z^\nu \equiv \dot{Z}^\nu.
\end{aligned} \quad (2.8)$$

We compute

$$\begin{aligned}
(D_X \nabla_{\dot{\gamma}} Y)^a &= \delta_X(\nabla_{\dot{\gamma}} Y)^a + \Gamma_{bc}^a X^b \nabla_{\dot{\gamma}} Y^c \\
&= \delta_X(\dot{\gamma}^j \partial_j Y^a + \Gamma_{bc}^a \dot{\gamma}^b Y^c) + \Gamma_{bc}^a X^b (\dot{\gamma}^j \partial_j Y^c + \Gamma_{ef}^c \dot{\gamma}^e Y^f) \\
&= \dot{X}^j \partial_j Y^a + \dot{\gamma}^j \partial_j \delta_X Y^a + \partial_m \Gamma_{bc}^a X^m \dot{\gamma}^b Y^c + \Gamma_{bc}^a \dot{X}^b Y^c + \Gamma_{bc}^a \dot{\gamma}^b \delta_X Y^c \\
&\quad + \Gamma_{bc}^a X^b \dot{\gamma}^j \partial_j Y^c + \Gamma_{bc}^a \Gamma_{ef}^c X^b \dot{\gamma}^e Y^f. \\
(\nabla_{\dot{\gamma}} D_X Y)^a &= \dot{\gamma}^j (\partial_j (D_X Y)^a + \Gamma_{bc}^a \dot{\gamma}^b (D_X Y)^c) \\
&= \dot{\gamma}^j \partial_j (\delta_X Y^a + \Gamma_{bc}^a X^b Y^c) + \Gamma_{bc}^a \dot{\gamma}^b (\delta_X Y^c + \Gamma_{sf}^c X^s Y^f) \\
&= \dot{\gamma}^j \partial_j \delta_X Y^a + \dot{\gamma}^j \partial_j \Gamma_{bc}^a X^b Y^c + \Gamma_{bc}^a \dot{X}^b Y^c + \Gamma_{bc}^a X^b \dot{Y}^c + \Gamma_{bc}^a \dot{\gamma}^b \delta_X Y^c \\
&\quad + \Gamma_{bc}^a \Gamma_{ef}^c \dot{\gamma}^b X^e Y^f.
\end{aligned}$$



Therefore

$$\begin{aligned}
(D_X \nabla_{\dot{\gamma}} Y - \nabla_{\dot{\gamma}} D_X Y)^a &= \partial_m \Gamma_{bc}^a X^m \dot{\gamma}^b Y^c - \partial_j \Gamma_{bc}^a \dot{\gamma}^j X^b Y^c + \Gamma_{bc}^a \Gamma_{ef}^c X^b \dot{\gamma}^e Y^f \\
&\quad - \Gamma_{bc}^a \Gamma_{ef}^c \dot{\gamma}^b X^e Y^f \\
&= (\partial_j \Gamma_{bc}^a - \partial_b \Gamma_{jc}^a + \Gamma_{je}^a \Gamma_{bc}^e - \Gamma_{be}^a \Gamma_{jc}^e) \dot{\gamma}^b X^j Y^c \\
&= R_{jbc}{}^a X^j \dot{\gamma}^b Y^c,
\end{aligned}$$

so

$$D_X \nabla_{\dot{\gamma}} Y - \nabla_{\dot{\gamma}} D_X Y = R(X, \dot{\gamma})Y.$$

□

**Corollary 2.5.** *At the loop  $\gamma$ ,  $[D_X, \Delta]Y = -\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}}Y$ . In particular,  $[D_X, \Delta]$  is a zeroth order operator.*

*Proof.*

$$\begin{aligned}
[D_X, \Delta]Y &= (-D_X \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} + \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} D_X)Y \\
&= -(\nabla_{\dot{\gamma}} D_X \nabla_{\dot{\gamma}} Y + R(X, \dot{\gamma})\nabla_{\dot{\gamma}} Y) + \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} D_X Y \\
&= -(\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} D_X Y + \nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) + R(X, \dot{\gamma})\nabla_{\dot{\gamma}} Y) + \nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} D_X Y \\
&= -\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}} Y.
\end{aligned}$$

□

Now we complete the proof of Theorem 2.2, showing in the process that  $A_X Y$  exists.

*Proof of Theorem 2.2.* By Proposition 2.3 and Corollary 2.5, we have

$$\begin{aligned}
\nabla_X^1 Y &= D_X Y + \frac{1}{2}(1 + \Delta)^{-1}[D_X, 1 + \Delta]Y + (X \leftrightarrow Y) + A_X Y \\
&= D_X Y + \frac{1}{2}(1 + \Delta)^{-1}(-\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}} Y) + (X \leftrightarrow Y) + A_X Y,
\end{aligned}$$

where  $(X \leftrightarrow Y)$  denotes the previous term with  $X$  and  $Y$  switched.

The curvature tensor satisfies

$$-\langle Z, R(X, Y)W \rangle = \langle R(X, Y)Z, W \rangle = \langle R(Z, W)X, Y \rangle$$

pointwise, so

$$\begin{aligned}
\langle A_X Y, Z \rangle_1 &= -\frac{1}{2} \langle [D_Z, 1 + \Delta] X, Y \rangle_0 \\
&= -\frac{1}{2} \langle (-\nabla_{\dot{\gamma}}(R(Z, \dot{\gamma})X) - R(Z, \dot{\gamma})\nabla_{\dot{\gamma}}X, Y) \rangle_0 \\
&= -\frac{1}{2} \langle R(Z, \dot{\gamma})X, \nabla_{\dot{\gamma}}Y \rangle_0 + \frac{1}{2} \langle R(Z, \dot{\gamma})\nabla_{\dot{\gamma}}X, Y \rangle_0 \\
&= -\frac{1}{2} \langle R(X, \nabla_{\dot{\gamma}}Y)Z, \dot{\gamma} \rangle_0 + \frac{1}{2} \langle R(\nabla_{\dot{\gamma}}X, Y)Z, \dot{\gamma} \rangle_0 \\
&= \frac{1}{2} \langle Z, R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} \rangle_0 - \frac{1}{2} \langle Z, R(\nabla_{\dot{\gamma}}X, Y)\dot{\gamma} \rangle_0 \\
&= \frac{1}{2} \langle Z, (1 + \Delta)^{-1}(R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} - R(\nabla_{\dot{\gamma}}X, Y)\dot{\gamma}) \rangle_1.
\end{aligned}$$

Thus  $A_X Y$  must equal  $\frac{1}{2}(1 + \Delta)^{-1}(R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} - R(\nabla_{\dot{\gamma}}X, Y)\dot{\gamma})$ . This makes sense: for  $X, Y \in H^{s'-1}$ ,  $A_X Y \in H^{s'} \subset H^1$ , since  $R$  is zeroth order.  $\square$

**Remark 2.4.** Locally on  $LM$ , we should have  $D_X Y = \delta_X^{LM} Y + \omega_X^{LM}(Y)$ . Now  $\delta_X^{LM} Y$  can only mean  $\frac{d}{d\tau}|_{\tau=0} \frac{d}{d\epsilon}|_{\epsilon=0} \gamma(\epsilon, \tau, \theta)$ , where  $\gamma(0, 0, \theta) = \gamma(\theta)$ ,  $\frac{d}{d\epsilon}|_{\epsilon=0} \gamma(\epsilon, 0, \theta) = X(\theta)$ ,  $\frac{d}{d\tau}|_{\tau=0} \gamma(\epsilon, \tau, \theta) = Y_{\gamma(\epsilon, 0, \cdot)}(\theta)$ . In other words,  $\delta_X^{LM} Y$  equals  $\delta_X Y$ . Since  $D_X Y^a = \delta_X Y^a + \Gamma_{bc}^a(\gamma(\theta))$ , the connection one-form  $\omega^{LM}$  for the  $L^2$  Levi-Civita connection on  $LM$  is related to the connection one-form  $\omega^M$  for the Levi-Civita connection on  $M$  by

$$\omega_X^{LM}(Y)^a(\gamma)(\theta) = \Gamma_{bc}^a(\gamma(\theta))X^b Y^c = \omega_X^M(Y)^a(\gamma(\theta)).$$

By this remark, we get

**Corollary 2.6.** *The connection one-form  $\omega^1$  for  $\nabla^1$  in exponential coordinates is*

$$\begin{aligned}
\omega_X^1(Y)(\gamma)(\theta) &= \omega_X^M(Y)(\gamma(\theta)) + \frac{1}{2} \{ (1 + \Delta)^{-1} [ -\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) - R(X, \dot{\gamma})\nabla_{\dot{\gamma}}Y \\
&\quad - \nabla_{\dot{\gamma}}(R(Y, \dot{\gamma})X) - R(Y, \dot{\gamma})\nabla_{\dot{\gamma}}X \\
&\quad + R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} - R(\nabla_{\dot{\gamma}}X, Y)\dot{\gamma} ] \}(\theta).
\end{aligned} \tag{2.9}$$

From this Corollary, we can directly compute the curvature of the  $s = 1$  metric connection. The result is unpleasant. Fortunately, for the characteristic classes discussed in [10], we only need the higher order symbols of the curvature. These are computed in Appendix A.

## 2.4. The Levi-Civita Connection for $s \in \mathbb{Z}^+$ .

For  $s > \frac{1}{2}$ , the proof of Prop. 2.3 extends directly to give

**Lemma 2.7.** *The Levi-Civita connection for the  $H^s$  metric is given by*

$$\nabla_X^s Y = D_X Y + \frac{1}{2}(1 + \Delta)^{-s} [D_X, (1 + \Delta)^s] Y + \frac{1}{2}(1 + \Delta)^{-s} [D_Y, (1 + \Delta)^s] X + A_X Y,$$

where we assume that for  $X, Y \in H^{s'-1}$ ,  $A_X Y \in H^s$  is characterized by

$$-\frac{1}{2}\langle [D_Z, (1 + \Delta)^s]X, Y \rangle_0 = \langle A_X Y, Z \rangle_s. \quad (2.10)$$

We now compute the bracket terms.

**Lemma 2.8.** *For  $s \in \mathbb{Z}^+$ , at the loop  $\gamma$ ,*

$$[D_X, (1 + \Delta)^s]Y = \sum_{k=1}^s (-1)^k \binom{s}{k} \sum_{j=0}^{2k-1} \nabla_{\dot{\gamma}}^j (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} Y). \quad (2.11)$$

*In particular,  $[D_X, (1 + \Delta)^s]Y$  is a  $\Psi$ DO of order at most  $2s - 1$  in either  $X$  or  $Y$ .*

*Proof.* The sum over  $k$  comes from the binomial expansion of  $(1 + \Delta)^s$ , so we just need an inductive formula for  $[D_X, \Delta^s]$ . The case  $s = 1$  is Proposition 2.3. For the induction step, we have

$$\begin{aligned} [D_X, \Delta^s] &= D_X \Delta^{s-1} \Delta - \Delta^s D_X \\ &= \Delta^{s-1} D_X \Delta + [D_X, \Delta^{s-1}] \Delta - \Delta^s D_X \\ &= \Delta^s D_X + \Delta^{s-1} [D_X, \Delta] + [D_X, \Delta^{s-1}] \Delta - \Delta^s D_X \\ &= \Delta^{s-1} (-\nabla_{\dot{\gamma}} (R(X, \dot{\gamma}) Y) - R(X, \dot{\gamma}) \nabla_{\dot{\gamma}} Y) \\ &\quad - \sum_{j=0}^{2s-3} (-1)^{s-1} \nabla_{\dot{\gamma}}^j (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-j-1} (-\nabla_{\dot{\gamma}}^2 Y)) \\ &= (-1)^{s-1} (-\nabla_{\dot{\gamma}}^{2s-1} (R(X, \dot{\gamma}) Y) - (-1)^{s-1} \nabla_{\dot{\gamma}}^{2s-2} (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}} Y)) \\ &\quad + \sum_{j=0}^{2s-3} (-1)^s \nabla_{\dot{\gamma}}^j (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-j-1} (-\nabla_{\dot{\gamma}}^2 Y)) \\ &= \sum_{j=0}^{2s-1} (-1)^s \nabla_{\dot{\gamma}}^j (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} Y). \end{aligned}$$

□

We check that  $A_X Y$  is a  $\Psi$ DO in  $X$  and  $Y$  for  $s \in \mathbb{Z}^+$ .

**Lemma 2.9.** *For  $s \in \mathbb{Z}^+$  and fixed  $X, Y \in H^{s'-1}$ ,  $A_X Y$  in (2.10) is an explicit  $\Psi$ DO in  $X$  and  $Y$  of order at most  $-1$ .*

*Proof.* By (2.11), for  $j, 2k-1-j \in \{0, 1, \dots, 2s-1\}$ , a typical term on the left hand side of (2.10) is

$$\begin{aligned}
\langle \nabla_{\dot{\gamma}}^j (R(Z, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} X), Y \rangle_0 &= (-1)^j \langle R(Z, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y \rangle_0 \\
&= (-1)^j \int_{S^1} g_{i\ell} (R(Z, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} X)^i (\nabla_{\dot{\gamma}}^j Y)^\ell d\theta \\
&= (-1)^j \int_{S^1} g_{i\ell} Z^k R_{krn}^i \dot{\gamma}^r (\nabla_{\dot{\gamma}}^{2k-1-j} X)^n (\nabla_{\dot{\gamma}}^j Y)^\ell d\theta \\
&= (-1)^j \int_{S^1} g_{tm} g^{kt} g_{i\ell} Z^m R_{krn}^i \dot{\gamma}^r (\nabla_{\dot{\gamma}}^{2k-1-j} X)^n (\nabla_{\dot{\gamma}}^j Y)^\ell d\theta \\
&= (-1)^j \langle Z, g^{kt} g_{i\ell} R_{krn}^i \dot{\gamma}^r (\nabla_{\dot{\gamma}}^{2k-1-j} X)^n (\nabla_{\dot{\gamma}}^j Y)^\ell \partial_t \rangle_0 \\
&= (-1)^j \langle Z, R_{rnl}^t \dot{\gamma}^r (\nabla_{\dot{\gamma}}^{2k-1-j} X)^n (\nabla_{\dot{\gamma}}^j Y)^\ell \partial_t \rangle_0 \\
&= (-1)^{j+1} \langle Z, R_{nlr}^t \dot{\gamma}^r (\nabla_{\dot{\gamma}}^{2k-1-j} X)^n (\nabla_{\dot{\gamma}}^j Y)^\ell \partial_t \rangle_0 \\
&= (-1)^{j+1} \langle Z, R(\nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y) \dot{\gamma} \rangle_0 \\
&= (-1)^{j+1} \langle Z, (1 + \Delta)^{-s} R(\nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y) \dot{\gamma} \rangle_s.
\end{aligned}$$

(In the integrals and inner products, the local expressions are in fact globally defined one-forms on  $S^1$ , resp. vector fields along  $\gamma$ , so we do not need a partition of unity.)  $(1 + \Delta)^{-s} R(\nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y) \dot{\gamma}$  is of order at most  $-1$  in either  $X$  or  $Y$ , so this term is in  $H^{s'} \subset H^s$ . Thus the last inner product is well defined.  $\square$

By (2.10), (2.11) and the proof of Lemma 2.9, we get

$$A_X Y = \sum_{k=1}^s (-1)^k \binom{s}{k} \sum_{j=0}^{2k-1} (-1)^{j+1} (1 + \Delta)^{-s} R(\nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y) \dot{\gamma}.$$

This gives:

**Theorem 2.10.** *For  $s \in \mathbb{Z}^+$ , the Levi-Civita connection for the  $H^s$  metric at the loop  $\gamma$  is given by*

$$\begin{aligned}
\nabla_X^s Y(\gamma) &= D_X Y(\gamma) + \frac{1}{2} (1 + \Delta)^{-s} \sum_{k=1}^s (-1)^k \binom{s}{k} \sum_{j=0}^{2k-1} \nabla_{\dot{\gamma}}^j (R(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^{2k-1-j} Y) \\
&\quad + (X \leftrightarrow Y) \\
&\quad + \sum_{k=1}^s (-1)^k \binom{s}{k} \sum_{j=0}^{2k-1} (-1)^{j+1} (1 + \Delta)^{-s} R(\nabla_{\dot{\gamma}}^{2k-1-j} X, \nabla_{\dot{\gamma}}^j Y) \dot{\gamma}.
\end{aligned}$$

## 2.5. The Levi-Civita Connection for General $s > \frac{1}{2}$ .

In this subsection, we show that the  $H^s$  Levi-Civita connection for general  $s > \frac{1}{2}$  exists in the strong sense of Remark 2.1. The formula is less explicit than in the  $s \in \mathbb{Z}^+$  case, but is good enough for symbol calculations.

By Lemma 2.7, we have to examine the term  $A_X Y$ , which, if it exists, is characterized by (2.10):

$$-\frac{1}{2}\langle [D_Z, (1 + \Delta)^s]X, Y \rangle_0 = \langle A_X Y, Z \rangle_s$$

for  $Z \in H^s$ . As explained in Remark 2.1, we may take  $X, Y \in H^{s'-1}$ . Throughout this section we assume that  $s' \gg s$ .

The following lemma extends Lemma 2.8.

**Lemma 2.11.** (i) For fixed  $Z \in H^{s'-1}$ ,  $[D_Z, (1 + \Delta)^s]X$  is a  $\Psi$ DO of order  $2s - 1$  in  $X$ . For  $\text{Re}(s) \neq 0$ , the principal symbol of  $[D_Z, (1 + \Delta)^s]$  is linear in  $s$ .

(ii) For fixed  $X \in H^{s'-1}$ ,  $[D_Z, (1 + \Delta)^s]X$  is a  $\Psi$ DO of order  $2s - 1$  in  $Z$ .

As usual, “of order  $2s - 1$ ” means “of order at most  $2s - 1$ .”

*Proof.* (i) For  $f : LM \rightarrow \mathbb{C}$ , we get  $[D_Z, (1 + \Delta)^s]fX = f[D_Z, (1 + \Delta)^s]X$ , since  $[f, (1 + \Delta)^s] = 0$ . Therefore,  $[D_Z, (1 + \Delta)^s]X$  depends only on  $X|_\gamma$ .

By Lemma 2.1,  $D_Z = \delta_Z + \Gamma \cdot Z$  in shorthand exponential coordinates. The Christoffel symbol term is zeroth order and  $(1 + \Delta)^s$  has scalar leading order symbol, so  $[\Gamma \cdot Z, (1 + \Delta)^s]$  has order  $2s - 1$ .

From the integral expression for  $(1 + \Delta)^s$ , it is immediate that

$$\begin{aligned} [\delta_Z, (1 + \Delta)^s]X &= (\delta_Z(1 + \Delta)^s)X + (1 + \Delta)^s\delta_Z X - (1 + \Delta)^s\delta_Z X \quad (2.12) \\ &= (\delta_Z(1 + \Delta)^s)X. \end{aligned}$$

$\delta_Z(1 + \Delta)^s$  is a limit of differences of  $\Psi$ DOs on bundles isomorphic to  $\gamma^*TM$ . Since the algebra of  $\Psi$ DOs is closed in the Fréchet topology of all  $C^k$  seminorms of symbols and smoothing terms on compact sets,  $\delta_Z(1 + \Delta)^s$  is a  $\Psi$ DO.

Since  $(1 + \Delta)^s$  has order  $2s$  and has scalar leading order symbol,  $[D_Z, (1 + \Delta)^s]$  have order  $2s - 1$ . For later purposes (§3.2), we compute some explicit symbols.

Assume  $\text{Re}(s) < 0$ . As in the construction of  $(1 + \Delta)^s$ , we will compute what the symbol asymptotics of  $\delta_Z(1 + \Delta)^s$  should be, and then construct an operator with these asymptotics. From the functional calculus for unbounded operators, we have

$$\begin{aligned} \delta_Z(1 + \Delta)^s &= \delta_Z \left( \frac{i}{2\pi} \int_\Gamma \lambda^s (1 + \Delta - \lambda)^{-1} d\lambda \right) \\ &= \frac{i}{2\pi} \int_\Gamma \lambda^s \delta_Z (1 + \Delta - \lambda)^{-1} d\lambda \quad (2.13) \\ &= -\frac{i}{2\pi} \int_\Gamma \lambda^s (1 + \Delta - \lambda)^{-1} (\delta_Z \Delta) (1 + \Delta - \lambda)^{-1} d\lambda, \end{aligned}$$

where  $\Gamma$  is a contour around the spectrum of  $1 + \Delta$ , and the hypothesis on  $s$  justifies the exchange of  $\delta_Z$  and the integral. The operator  $A = (1 + \Delta - \lambda)^{-1} \delta_Z \Delta (1 + \Delta - \lambda)^{-1}$

is a  $\Psi$ DO of order  $-3$  with top order symbol

$$\begin{aligned}\sigma_{-3}(A)(\theta, \xi)_j^\ell &= (\xi^2 - \lambda)^{-1} \delta_k^\ell (-2Z^i \partial_i \Gamma_{\nu\mu}^k \dot{\gamma}^\nu - 2\Gamma_{\nu\mu}^k \dot{Z}^\nu) \xi (\xi^2 - \lambda)^{-1} \delta_j^\mu \\ &= (-2Z^i \partial_i \Gamma_{\nu j}^\ell \dot{\gamma}^\nu - 2\Gamma_{\nu j}^\ell \dot{Z}^\nu) \xi (\xi^2 - \lambda)^{-2}.\end{aligned}$$

Thus the top order symbol of  $\delta_Z(1 + \Delta)^s$  should be

$$\begin{aligned}\sigma_{2s-1}(\delta_Z(1 + \Delta)^s)(\theta, \xi)_j^\ell &= -\frac{i}{2\pi} \int_\Gamma \lambda^s (-2Z^i \partial_i \Gamma_{\nu j}^\ell \dot{\gamma}^\nu - 2\Gamma_{\nu j}^\ell \dot{Z}^\nu) \xi (\xi^2 - \lambda)^{-2} d\lambda \\ &= \frac{i}{2\pi} \int_\Gamma s \lambda^{s-1} (-2Z^i \partial_i \Gamma_{\nu j}^\ell \dot{\gamma}^\nu - 2\Gamma_{\nu j}^\ell \dot{Z}^\nu) \xi (\xi^2 - \lambda)^{-1} d\lambda \\ &= s (-2Z^i \partial_i \Gamma_{\nu j}^\ell \dot{\gamma}^\nu - 2\Gamma_{\nu j}^\ell \dot{Z}^\nu) \xi (\xi^2 - \lambda)^{s-1}.\end{aligned}\quad (2.14)$$

Similarly, all the terms in the symbol asymptotics for  $A$  are of the form  $B_j^\ell \xi^n (\xi^2 - \lambda)^m$  for some matrices  $B_j^\ell = B_j^\ell(n, m)$ . This produces a symbol sequence  $\sum_{k \in \mathbb{Z}^+} \sigma_{2s-k}$ , and there exists a  $\Psi$ DO  $P$  with  $\sigma(P) = \sum \sigma_{2s-k}$ . (As in §2.2, we first produce operators  $P_i$  on a coordinate cover  $U_i$  of  $S^1$ , and then set  $P = \sum_i \phi_i P_i \psi_i$ .) The construction depends on the choice of local coordinates covering  $\gamma$ , the partition of unity and cutoff functions as above, and a cutoff function in  $\xi$ ; as usual, different choices change the operator by a smoothing operator. Standard estimates show that  $P - \delta_Z(1 + \Delta)^s$  is a smoothing operator, which verifies explicitly that  $\delta_Z(1 + \Delta)^s$  is a  $\Psi$ DO of order  $2s - 1$ .

For  $\text{Re}(s) > 0$ , motivated by differentiating  $(1 + \Delta)^{-s} \circ (1 + \Delta)^s = \text{Id}$ , we set

$$\delta_Z(1 + \Delta)^s = -(1 + \Delta)^s \circ \delta_Z(1 + \Delta)^{-s} \circ (1 + \Delta)^s. \quad (2.15)$$

This is again a  $\Psi$ DO of order  $2s - 1$  with principal symbol linear in  $s$ .

(ii) As a  $\Psi$ DO of order  $2s$ ,  $(1 + \Delta)^s$  has the expression

$$(1 + \Delta)^s X(\gamma)(\theta) = \int_{T^*S^1} e^{i(\theta - \theta') \cdot \xi} p(\theta, \xi) X(\gamma)(\theta') d\theta' d\xi,$$

where we omit the cover of  $S^1$  and its partition of unity on the right hand side. Here  $p(\theta, \xi)$  is the symbol of  $(1 + \Delta)^s$ , which has the asymptotic expansion

$$p(\theta, \xi) \sim \sum_{k=0}^{\infty} p_{2s-k}(\theta, \xi).$$

The covariant derivative along  $\gamma$  on  $Y \in \Gamma(\gamma^*TM)$  is given by

$$\begin{aligned}\frac{DY}{d\gamma} &= (\gamma^* \nabla^M)_{\partial_\theta}(Y) = \partial_\theta Y + (\gamma^* \omega^M)(\partial_\theta)(Y) \\ &= \partial_\theta(Y^i) \partial_i + \dot{\gamma}^t Y^r \Gamma_{tr}^j \partial_j,\end{aligned}$$

where  $\nabla^M$  is the Levi-Civita connection on  $M$  and  $\omega^M$  is the connection one-form in exponential coordinates on  $M$ . For  $\Delta = (\frac{D}{d\gamma})^* \frac{D}{d\gamma}$ , an integration by parts using the

formula  $\partial_t g_{ar} = \Gamma_{\ell t}^n g_{rn} + \Gamma_{rt}^n g_{\ell n}$  gives

$$(\Delta Y)^k = -\partial_\theta^2 Y^k - 2\Gamma_{\nu\mu}^k \dot{\gamma}^\nu \partial_\theta Y^\mu - (\partial_\theta \Gamma_{\nu\delta}^k \dot{\gamma}^\nu + \Gamma_{\nu\delta}^k \ddot{\gamma}^\nu + \Gamma_{\nu\mu}^k \Gamma_{\varepsilon\delta}^\mu \dot{\gamma}^\varepsilon \dot{\gamma}^\nu) Y^\delta.$$

Thus  $p_{2s}(\theta, \xi) = |\xi|^2$  is independent of  $\gamma$ , but the lower order symbols depend on derivatives of both  $\gamma$  and the metric on  $M$ .

We have

$$[D_Z, (1 + \Delta)^s]X(\gamma)(\theta) = D_Z \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) X(\gamma)(\theta') d\theta' d\xi \quad (2.16)$$

$$- \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) D_Z X(\gamma)(\theta') d\theta' d\xi. \quad (2.17)$$

In local coordinates, (2.16) equals

$$\begin{aligned} & \left[ D_Z \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) X(\gamma)(\theta') d\theta' d\xi \right]^a \\ &= \delta_Z \left[ \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) X(\gamma)(\theta') d\theta' d\xi \right]^a (\theta) \\ & \quad + \Gamma_{bc}^a Z^b(\gamma)(\theta) \left[ \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) X(\gamma)(\theta') d\theta' d\xi \right]^c (\theta). \end{aligned} \quad (2.18)$$

Here we have suppressed matrix indices in  $p$  and  $X$ . We can bring  $\delta_Z$  past the integral on the right hand side of (2.18). If  $\gamma_\epsilon$  is a family of curves with  $\gamma_0 = \gamma$ ,  $\dot{\gamma}_\epsilon = Z$ , then

$$\delta_Z p(\theta, \xi) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} p(\gamma_\epsilon, \theta, \xi) = \left. \frac{d\gamma_\epsilon^k}{d\epsilon} \right|_{\epsilon=0} \partial_k p(\gamma, \theta, \xi) = Z^k(\gamma(\theta)) \cdot \partial_k p(\theta, \xi).$$

Substituting this into (2.18) gives

$$\begin{aligned} & \left[ D_Z \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) X(\gamma)(\theta') d\theta' d\xi \right]^a \\ &= \left[ \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} Z^k(\gamma)(\theta) \cdot \partial_k p(\theta, \xi) X(\gamma)(\theta') d\theta' d\xi \right]^a \\ & \quad + \Gamma_{bc}^a Z^b(\gamma)(\theta) \left[ \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) X(\gamma)(\theta') d\theta' d\xi \right]^c (\theta) \\ & \quad + \left[ \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) \delta_Z X(\gamma)(\theta') d\theta' d\xi \right]^a (\theta). \end{aligned} \quad (2.19)$$

Similarly, (2.17) equals

$$\begin{aligned} & \left[ \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) D_Z X(\gamma)(\theta') d\theta' d\xi \right]^a \\ &= \left[ \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) \delta_Z X(\gamma)(\theta') d\theta' d\xi \right]^a \\ & \quad + \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) {}_e\Gamma_{bc}^e Z^b(\gamma)(\theta') X^c(\gamma)(\theta') d\theta' d\xi. \end{aligned} \quad (2.20)$$

Substituting (2.19), (2.20), into (2.16), (2.17), respectively, gives

$$\begin{aligned} & ([D_Z, (1 + \Delta)^s] X(\theta))^a \\ &= Z^b(\theta) \cdot \left[ \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} (\partial_b p_e^a(\theta, \xi) + \Gamma_{bc}^a(\gamma(\theta)) p_e^c(\theta, \xi)) X^e(\theta') d\theta' d\xi \right] \\ & \quad - \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} p(\theta, \xi) {}_e\Gamma_{bc}^e(\gamma(\theta')) Z^b(\theta') X^c(\theta') d\theta' d\xi, \end{aligned} \quad (2.21)$$

where  $X(\theta') = X(\gamma)(\theta')$  and similarly for  $Z$ .

The first term on the right hand side of (2.21) is order zero in  $Z$ ; note that  $0 < 2s-1$ , since  $s > \frac{1}{2}$ . For the last term in (2.21), we do a change of variables typically used in the proof that the composition of  $\Psi$ DOs is a  $\Psi$ DO. Set

$$q(\theta, \theta', \xi)_b^a = p(\theta, \xi) {}_e\Gamma_{bc}^e(\gamma(\theta')) X^c(\theta'), \quad (2.22)$$

so the last term equals

$$\begin{aligned} (PZ)^a(\theta) &\stackrel{\text{def}}{=} \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} q(\theta, \theta', \xi)_b^a Z^b(\theta') d\theta' d\xi \\ &= \int_{T^*S^1} e^{i(\theta-\theta')\cdot\xi} q(\theta, \theta', \xi)_b^a e^{i(\theta'-\theta'')\cdot\eta} Z^b(\theta'') d\theta'' d\eta d\theta' d\xi, \end{aligned}$$

by applying Fourier transform and its inverse to  $Z$ . A little algebra gives

$$(PZ)^a(\theta) = \int_{T^*S^1} e^{i(\theta-\theta')\cdot\eta} r(\theta, \eta)_b^a Z^b(\theta') d\theta' d\eta, \quad (2.23)$$

with

$$\begin{aligned} r(\theta, \eta) &= \int_{T^*S^1} e^{i(\theta-\theta')\cdot(\xi-\eta)} q(\theta, \theta', \xi) d\theta' d\xi \\ &= \int_{T^*S^1} e^{it\cdot\xi} q(\theta, \theta-t, \eta+\xi) dt d\xi. \end{aligned}$$

In the last line we continue to abuse notation by treating the integral in local coordinates in  $t = \theta - \theta'$  lying in an interval  $I \subset \mathbb{R}$  and implicitly summing over a cover and partition of unity of  $S^1$ ; thus we can consider  $q$  as a compactly supported function



in  $t \in \mathbb{R}$ . Substituting in the Taylor expansion of  $q(\theta, \theta - t, \eta + \xi)$  in  $\xi$  gives in local coordinates

$$\begin{aligned} r(\theta, \eta) &= \int_{T^*\mathbb{R}} e^{it \cdot \xi} \left[ \sum_{\alpha, |\alpha|=0}^N \frac{1}{\alpha!} \partial_\xi^\alpha |_{\xi=0} q(\theta, \theta - t, \eta + \xi) \xi^\alpha + O(|\xi|^{N+1}) \right] dt d\xi \\ &= \sum_{\alpha, |\alpha|=0}^N \frac{i^{|\alpha|}}{\alpha!} \partial_t^\alpha \partial_\xi^\alpha q(\theta, \theta, \eta) + O(|\xi|^{N+1}). \end{aligned} \quad (2.24)$$

Thus  $P$  in (2.23) is a  $\Psi$ DO with apparent top order symbol  $q(\theta, \theta, \eta)$ , which by (2.22) has order  $2s$ . The top order symbol can be computed in any local coordinates on  $S^1$  and  $\gamma^*TM$ . If we choose manifold coordinates (see §2.3) which are Riemannian normal coordinates centered at  $\gamma(\theta)$ , the Christoffel symbols vanish at this point, and so

$$q(\theta, \theta, \eta)_b^a = p(\theta, \xi)_e^a \Gamma_{bc}^e(\gamma(\theta)) X^c(\theta) = 0.$$

Thus  $P$  is in fact of order  $2s - 1$ , and so both terms on the right hand side of (2.21) have order at most  $2s - 1$ . □

**Remark 2.5.** (i) For  $s \in \mathbb{Z}^+$ ,  $\delta_Z(1 + \Delta)^s$  differs from the usual definition by a smoothing operator.

(ii) For all  $s$ , the proof of Lemma 2.11(i) shows that for all  $k$ ,  $\sigma_k(\delta_Z(1 + \Delta)^s) = \delta_Z(\sigma_k((1 + \Delta)^s))$  in local coordinates.

We can now complete the computation of the Levi-Civita connection for general  $s$ .

Let  $[D, (1 + \Delta)^s]X^*$  be the formal  $L^2$  adjoint of  $[D, (1 + \Delta)^s]X$ . We abbreviate  $[D, (1 + \Delta)^s]X^*(Y)$  by  $[D_Y, (1 + \Delta)^s]X^*$ .

**Theorem 2.12.** (i) For  $s > \frac{1}{2}$ , The Levi-Civita connection for the  $H^s$  metric is given by

$$\begin{aligned} \nabla_X^s Y &= D_X Y + \frac{1}{2}(1 + \Delta)^{-s} [D_X, (1 + \Delta)^s] Y + \frac{1}{2}(1 + \Delta)^{-s} [D_Y, (1 + \Delta)^s] X \\ &\quad - \frac{1}{2}(1 + \Delta)^{-s} [D_Y, (1 + \Delta)^s] X^*. \end{aligned} \quad (2.25)$$

(ii) The connection one-form  $\omega^s$  in exponential coordinates is given by

$$\begin{aligned} \omega_X^s(Y)(\gamma)(\theta) &= \omega^M(Y)(\gamma(\theta)) + \left( \frac{1}{2}(1 + \Delta)^{-s} [D_X, (1 + \Delta)^s] Y + \frac{1}{2}(1 + \Delta)^{-s} [D_Y, (1 + \Delta)^s] X \right. \\ &\quad \left. - \frac{1}{2}(1 + \Delta)^{-s} [D_Y, (1 + \Delta)^s] X^* \right) (\gamma)(\theta). \end{aligned} \quad (2.26)$$

(iii) The connection one-form takes values in zeroth order  $\Psi$ DOs.

*Proof.* Since  $[D_Z, (1 + \Delta)^s]X$  is a  $\Psi$ DO in  $Z$  of order  $2s - 1$ , its formal adjoint is a  $\Psi$ DO of the same order. Thus

$$\langle [D_Z, (1 + \Delta)^s]X, Y \rangle_0 = \langle Z, [D_\cdot, (1 + \Delta)^s]X^*(Y) \rangle = \langle Z, (1 + \Delta)^{-s}[D_Y, (1 + \Delta)^s]X^* \rangle_s.$$

Thus  $A_X Y$  in (2.10) satisfies  $A_X Y = (1 + \Delta)^{-s}[D_Y, (1 + \Delta)^s]X^*$ . Lemma 2.7 applies to all  $s > \frac{1}{2}$ , so (i) follows. (ii) follows as in Corollary 2.6. Since  $\omega^M$  is zeroth order and all other terms have order  $-1$ , (iii) holds as well.  $\square$

**Remark 2.6.** This theorem implies that the Levi-Civita connection exists for the  $H^s$  metric in the strong sense: for  $X \in T_\gamma LM = H^{s'-1}(\gamma^* TM)$  and  $Y_\eta \in H^{s'-1}(\eta^* TM)$  a smooth vector field on  $LM = H^{s'}(S^1, M)$ ,  $\nabla_X^s Y(\gamma) \in H^{s'-1}(\gamma^* TM)$ . (See Remark 2.1.) For each term except  $D_X Y$  on the right hand side of (2.25) is order  $-1$  in  $Y$ , and so takes  $H^{s'-1}$  to  $H^{s'} \subset H^{s'-1}$ . For  $D_X Y = \delta_X Y + \Gamma \cdot Y$ ,  $\Gamma$  is zeroth order and so bounded on  $H^{s'-1}$ . Finally, the definition of a smooth vector field on  $LM$  implies that  $\delta_X Y$  stays in  $H^{s'-1}$  for all  $X$ .

**2.6. Extensions of the Frame Bundle of  $LM$ .** In this subsection we discuss the choice of structure group for the  $H^s$  and Levi-Civita connections on  $LM$ .

Let  $\mathcal{H}$  be the Hilbert space  $H^{s_0}(\gamma^* TM)$  for a fixed  $s_0$  and  $\gamma$ . Let  $GL(\mathcal{H})$  be the group of bounded invertible linear operators on  $\mathcal{H}$ ; inverses of elements are bounded by the closed graph theorem.  $GL(\mathcal{H})$  has the subset topology of the norm topology on  $\mathcal{B}(\mathcal{H})$ , the bounded linear operators on  $\mathcal{H}$ .  $GL(\mathcal{H})$  is an infinite dimensional Banach Lie group, as a group which is an open subset of the infinite dimensional Hilbert manifold  $\mathcal{B}(\mathcal{H})$  [11, p. 59], and has Lie algebra  $\mathcal{B}(\mathcal{H})$ . Let  $\Psi DO_{\leq 0}, \Psi DO_0^*$  denote the algebra of classical  $\Psi$ DOs of nonpositive order and the group of invertible zeroth order  $\Psi$ DOs, respectively, where all  $\Psi$ DOs act on  $\mathcal{H}$ . Note that  $\Psi DO_0^* \subset GL(\mathcal{H})$ .

**Remark 2.7.** The inclusions of  $\Psi DO_0^*, \Psi DO_{\leq 0}$  into  $GL(\mathcal{H}), \mathcal{B}(\mathcal{H})$  are trivially continuous in the subset topology. For the Fréchet topology on  $\Psi DO_{\leq 0}$ , the inclusion is continuous as in [9].

We recall the relationship between the connection one-form  $\alpha_{FN}$  on the frame bundle  $FN$  of a manifold  $N$  and local expressions for the connection on  $TN$ . For  $U \subset N$ , let  $\chi : U \rightarrow FN$  be a local section. A metric connection  $\nabla$  on  $TN$  with local connection one-form  $\omega$  determines a connection  $\alpha_{FN} \in \Lambda^1(FN, \mathfrak{o}(n))$  on  $FN$  by (i)  $\alpha_{FN}$  is the Maurer-Cartan one-form on each fiber, and (ii)  $\alpha_{FN}(Y_u) = \omega(X_p)$ , for  $Y_u = \chi_* X_p$  [15, Ch. 8, Vol. II], or equivalently  $\chi^* \alpha_{FN} = \omega$ .

This applies to  $N = LM$ . The frame bundle  $FLM \rightarrow LM$  is constructed as in the finite dimensional case. The fiber over  $\gamma$  is isomorphic to the gauge group  $\mathcal{G}$  of  $\mathcal{R}$  and fibers are glued by the transition functions for  $TLM$ . Thus the frame bundle is topologically a  $\mathcal{G}$ -bundle.

However, by Theorem 2.12, the Levi-Civita connection one-form  $\omega_X^s$  takes values in  $\Psi DO_{\leq 0}$ . The curvature two-form  $\Omega^s = d_{LM}\omega^s + \omega^s \wedge \omega^s$  also takes values in  $\Psi DO_{\leq 0}$ . (Here  $d_{LM}\omega^s(X, Y)$  is defined by the Cartan formula for the exterior derivative.)

These forms should take values in the Lie algebra of the structure group. Thus we should extend the structure group to the Fréchet Lie group  $\Psi\text{DO}_0^*$ , since its Lie algebra is  $\Psi\text{DO}_{\leq 0}$  [12]. This leads to an extended frame bundles, also denoted  $FLM$ . The transition functions are unchanged, since  $\mathcal{G} \subset \Psi\text{DO}_0^*$ . Thus  $(FLM, \alpha^s = \alpha_{FLM}^s)$  as a geometric bundle (i.e. as a bundle with connection  $\alpha^s$  associated to  $\nabla^s$ ) is a  $\Psi\text{DO}_0^*$ -bundle.

In summary, for the Levi-Civita connections we have

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & FLM \\ & \downarrow & \\ & LM & \end{array} \quad \begin{array}{ccc} \Psi\text{DO}_0^* & \longrightarrow & (FLM, \alpha^s) \\ & \downarrow & \\ & LM & \end{array}$$

**Remark 2.8.** If we extend the structure group of the frame bundle with connection from  $\Psi\text{DO}_0^*$  to  $GL(\mathcal{H})$ , the frame bundle becomes trivial by Kuiper's theorem. Thus there is a potential loss of information if we pass to the larger frame bundle.

The situation is similar to the following examples. Let  $E \rightarrow S^1$  be the  $GL(1, \mathbb{R})$  (real line) bundle with gluing functions (multiplication by) 1 at  $1 \in S^1$  and 2 at  $-1 \in S^1$ .  $E$  is trivial as a  $GL(1, \mathbb{R})$ -bundle, with global section  $f$  with  $\lim_{\theta \rightarrow -\pi^+} f(e^{i\theta}) = 1$ ,  $f(1) = 1$ ,  $\lim_{\theta \rightarrow \pi^-} f(e^{i\theta}) = 1/2$ . However, as a  $GL(1, \mathbb{Q})^+$ -bundle,  $E$  is nontrivial, as a global section is locally constant. As a second example, let  $E \rightarrow M$  be a nontrivial  $GL(n, \mathbb{C})$ -bundle. Embed  $\mathbb{C}^n$  into a Hilbert space  $\mathcal{H}$ , and extend  $E$  to an  $GL(\mathcal{H})$ -bundle  $\mathcal{E}$  with fiber  $\mathcal{H}$  and with the transition functions for  $E$  (extended by the identity in directions perpendicular to the image of  $E$ ). Then  $\mathcal{E}$  is trivial.

### 3. The Loop Group Case

In this section, we relate our work to Freed's work on based loop groups  $\Omega G$  [3]. We find a particular representation of the loop algebra that controls the order of the curvature of the  $H^1$  metric on  $\Omega G$ .

$\Omega G \subset LG$  has tangent space  $T_\gamma \Omega G = \{X \in T_\gamma LG : X(0) = X(2\pi) = 0\}$  in some Sobolev topology. Instead of using  $D^2/d\gamma^2$  to define the Sobolev spaces, the usual choice is  $\Delta_{S^1} = -d^2/d\theta^2$  coupled to the identity operator on the Lie algebra  $\mathfrak{g}$ . Since this operator has no kernel on  $T_\gamma \Omega M$ ,  $1 + \Delta$  is replaced by  $\Delta$ . These changes in the  $H^s$  inner product do not alter the spaces of Sobolev sections, but the  $H^s$  metrics on  $\Omega G$  are no longer induced from a metric on  $G$  as in the previous sections.

This simplifies the calculations of the Levi-Civita connections. In particular,  $[D_Z, \Delta^s] = 0$ , so there is no term  $A_X Y$  as in (2.10). As a result, one can work directly with the six term formula (2.3). For  $X, Y, Z$  left invariant vector fields, the first three terms on the right hand side of (2.3) vanish. Under the standing assumption that  $G$  has a left invariant, Ad-invariant inner product, one obtains

$$2\nabla_X^{(s)} Y = [X, Y] + \Delta^{-s}[X, \Delta^s Y] + \Delta^{-s}[Y, \Delta^s X]$$

[3].

It is an interesting question to compute the order of the curvature operator as a function of  $s$ . For based loops, Freed proved that this order is at most  $-1$ . In [8], it is shown that the order of  $\Omega^s$  is at most  $-2$  for all  $s \neq 1/2, 1$  on both  $\Omega G$  and  $LG$ , and is exactly  $-2$  for  $G$  nonabelian. For the case  $s = 1$ , we have a much stronger result.

**Proposition 3.1.** *The curvature of the Levi-Civita connection for the  $H^1$  inner product on  $\Omega G$  associated to  $-\frac{d^2}{d\theta^2} \otimes \text{Id}$  is a  $\Psi\text{DO}$  of order  $-\infty$ .*

PROOF: We give two quite different proofs.

By [3], the  $s = 1$  curvature operator  $\Omega = \Omega^1$  satisfies

$$\langle \Omega(X, Y)Z, W \rangle_1 = \left( \int_{S^1} [Y, \dot{Z}], \int_{S^1} [X, \dot{W}] \right)_{\mathfrak{g}} - (X \leftrightarrow Y),$$

where the inner product is the Ad-invariant form on the Lie algebra  $\mathfrak{g}$ . We want to write the right hand side of this equation as an  $H^1$  inner product with  $W$ , in order to recognize  $\Omega(X, Y)$  as a  $\Psi\text{DO}$ .

Let  $\{e_i\}$  be an orthonormal basis of  $\mathfrak{g}$ , considered as a left-invariant frame of  $TG$  and as global sections of  $\gamma^*TG$ . Let  $c_{ij}^k = ([e_i, e_j], e_k)_{\mathfrak{g}}$  be the structure constants of  $\mathfrak{g}$ . (The Levi-Civita connection on left invariant vector fields for the left invariant metric is given by  $\nabla_X Y = \frac{1}{2}[X, Y]$ , so the structure constants are twice the Christoffel symbols.) For  $X = X^i e_i = X^i(\theta) e_i, Y = Y^j e_j$ , etc., integration by parts gives

$$\langle \Omega(X, Y)Z, W \rangle_1 = \left( \int_{S^1} \dot{Y}^i Z^j d\theta \right) \left( \int_{S^1} \dot{X}^\ell W^m d\theta \right) c_{ij}^k c_{\ell m}^n \delta_{kn} - (X \leftrightarrow Y).$$

Since

$$\int_{S^1} c_{\ell m}^n \dot{X}^\ell W^m = \int_{S^1} \left( \delta^{mc} c_{\ell c}^n \dot{X}^\ell e_m, W^b e_b \right)_{\mathfrak{g}} = \left\langle \Delta^{-1}(\delta^{mc} c_{\ell c}^n \dot{X}^\ell e_m), W \right\rangle_1,$$

we get

$$\begin{aligned} \langle \Omega(X, Y)Z, W \rangle_1 &= \left\langle \left[ \int_{S^1} \dot{Y}^i Z^j \right] c_{ij}^k \delta_{kn} \delta^{ms} c_{\ell s}^n \Delta^{-1}(\dot{X}^\ell e_m), W \right\rangle_1 - (X \leftrightarrow Y) \\ &= \left\langle \left[ \int_{S^1} a_j^k(\theta, \theta') Z^j(\theta') d\theta' \right] e_k, W \right\rangle_1, \end{aligned}$$

with

$$a_j^k(\theta, \theta') = \dot{Y}^i(\theta') c_{ij}^r \delta_{rn} \delta^{ms} c_{\ell s}^n \left( \Delta^{-1}(\dot{X}^\ell e_m) \right)^k(\theta) - (X \leftrightarrow Y). \quad (3.1)$$

We now show that  $Z \mapsto \left( \int_{S^1} a_j^k(\theta, \theta') Z^j(\theta') d\theta' \right) e_k$  is a smoothing operator. Applying Fourier transform and Fourier inversion to  $Z^j$  yields

$$\begin{aligned} \int_{S^1} a_j^k(\theta, \theta') Z^j(\theta') d\theta' &= \int_{S^1 \times \mathbb{R} \times S^1} a_j^k(\theta, \theta') e^{i(\theta' - \theta'') \cdot \xi} Z^j(\theta'') d\theta'' d\xi d\theta' \\ &= \int_{S^1 \times \mathbb{R} \times S^1} \left[ a_j^k(\theta, \theta') e^{-i(\theta - \theta') \cdot \xi} \right] e^{i(\theta - \theta'') \cdot \xi} Z^j(\theta'') d\theta'' d\xi d\theta', \end{aligned}$$

so  $\Omega(X, Y)$  is a  $\Psi$ DO with symbol

$$b_j^k(\theta, \xi) = \int_{S^1} a_j^k(\theta, \theta') e^{i(\theta - \theta') \cdot \xi} d\theta', \quad (3.2)$$

with the usual mixing of local and global notation.

For fixed  $\theta$ , (3.2) contains the Fourier transform of  $\dot{Y}^i(\theta')$  and  $\dot{X}^i(\theta')$ , as these are the only  $\theta'$ -dependent terms in (3.1). Since the Fourier transform is taken in a local chart with respect to a partition of unity, and since in each chart  $\dot{Y}^i$  and  $\dot{X}^i$  times the partition of unity function is compactly supported, the Fourier transform of  $a_j^k$  in each chart is rapidly decreasing. Thus  $b_j^k(\theta, \xi)$  is the product of a rapidly decreasing function with  $e^{i\theta \cdot \xi}$ , and hence is of order  $-\infty$ .

We now give a second proof. For all  $s$ ,

$$\nabla_X Y = \frac{1}{2}[X, Y] - \frac{1}{2}\Delta^{-s}[\Delta^s X, Y] + \frac{1}{2}\Delta^{-s}[X, \Delta^s Y].$$

Label the terms on the right hand side (1) – (3). As an operator on  $Y$  for fixed  $X$ , the symbol of (1) is  $\sigma((1))_\mu^a = \frac{1}{2}X^\varepsilon c_{\varepsilon\mu}^a$ . Abbreviating  $(\xi^2)^{-s}$  by  $\xi^{-2s}$ , we have

$$\begin{aligned} \sigma((2))_\mu^a &\sim -\frac{1}{2}c_{\varepsilon\mu}^a \left[ \xi^{-2s} \Delta^s X^\varepsilon - \frac{2s}{i} \xi^{-2s-1} \partial_\theta \Delta^s X^\varepsilon \right. \\ &\quad \left. + \sum_{\ell=2}^{\infty} \frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell \ell!} \xi^{-2s-\ell} \partial_\theta^\ell \Delta^s X^\varepsilon \right] \\ \sigma((3))_\mu^a &\sim \frac{1}{2}c_{\varepsilon\mu}^a \left[ X^\varepsilon + \sum_{\ell=1}^{\infty} \frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell \ell!} \xi^{-\ell} \partial_\theta^\ell X^\varepsilon \right]. \end{aligned}$$

Thus

$$\begin{aligned} \sigma(\nabla_X)_\mu^a &\sim \frac{1}{2}c_{\varepsilon\mu}^a \left[ 2X^\varepsilon - \xi^{-2s} \Delta^s X^\varepsilon + \frac{2s}{i} \xi^{-2s-1} \partial_\theta \Delta^s X^\varepsilon \right. \\ &\quad \left. - \sum_{\ell=2}^{\infty} \frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell \ell!} \xi^{-2s-\ell} \partial_\theta^\ell \Delta^s X^\varepsilon \right. \\ &\quad \left. + \sum_{\ell=1}^{\infty} \frac{(-2s)(-2s-1)\dots(-2s-\ell+1)}{i^\ell \ell!} \xi^{-\ell} \partial_\theta^\ell X^\varepsilon \right]. \end{aligned} \quad (3.3)$$

Set  $s = 1$  in (3.3), and replace  $\ell$  by  $\ell - 2$  in the first infinite sum. Since  $\Delta = -\partial_\theta^2$ , a little algebra gives

$$\sigma(\nabla_X)_\mu^a \sim c_{\varepsilon\mu}^a \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{i^\ell} \partial_\theta^\ell X^\varepsilon \xi^{-\ell} = \text{ad} \left( \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{i^\ell} \partial_\theta^\ell X \xi^{-\ell} \right). \quad (3.4)$$

Denote the infinite sum in the last term of (3.4) by  $W(X, \theta, \xi)$ . The map  $X \mapsto W(X, \theta, \xi)$  takes the Lie algebra of left invariant vector fields on  $LG$  to the Lie algebra

$L\mathfrak{g}[[\xi^{-1}]]$ , the space of formal  $\Psi$ DOs of nonpositive integer order on the trivial bundle  $S^1 \times \mathfrak{g} \rightarrow S^1$ , where the Lie bracket on the target involves multiplication of power series and bracketing in  $\mathfrak{g}$ . We claim that this map is a Lie algebra homomorphism. Assuming this, we see that

$$\begin{aligned} \sigma(\Omega(X, Y)) &= \sigma([\nabla_X, \nabla_Y] - \nabla_{[X, Y]}) \sim \sigma([\text{ad } W(X), \text{ad } W(Y)] - \text{ad } W([X, Y])) \\ &= \sigma(\text{ad}([W(X), W(Y)]) - \text{ad } W([X, Y])) = 0, \end{aligned}$$

which proves that  $\Omega(X, Y)$  is a smoothing operator.

To prove the claim, set  $X = x_n^a e^{in\theta} e_a$ ,  $Y = y_m^b e^{im\theta} e_b$ . Then

$$\begin{aligned} W([X, Y]) &= W(x_n^a y_m^b e^{i(n+m)\theta} c_{ab}^k e_k) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{i^\ell} c_{ab}^k \partial_\theta^\ell (x_n^a y_m^b e^{i(n+m)\theta}) \xi^{-\ell} e_k \\ [W(X), W(Y)] &= \sum_{\ell=0}^{\infty} \sum_{p+q=\ell} \frac{(-1)^{p+q}}{i^{p+q}} \partial_\theta^p (x_n^a e^{in\theta}) \partial_\theta^q (y_m^b e^{im\theta}) \xi^{-(p+q)} c_{ab}^k e_k, \end{aligned}$$

and these two sums are clearly equal.  $\square$

It would be interesting to understand how the map  $W$  fits into the representation theory of the loop algebra  $L\mathfrak{g}$ .

## APPENDIX A. Local Symbol Calculations

We compute the 0 and  $-1$  order symbols of the connection one-form  $\omega^1$  and the curvature two-form  $\Omega^1$  of the  $s = 1$  Levi-Civita connection. We also compute the 0 and  $-1$  order symbols of the connection one-form for the general  $s > \frac{1}{2}$  connection, and the 0 order symbol of the curvature of the general  $s$  connection. The formulas show that the  $s$ -dependence of these symbols is linear, which will be used to define regularized (i.e.  $s$ -independent) Wodzicki-Chern-Simons classes in [10].

### A.1. Connection and Curvature Symbols for $s = 1$ .

Using Corollary 2.6, we can compute these symbols easily in manifold coordinates.

**Lemma A.1.** (i) At  $\gamma(\theta)$ ,  $\sigma_0(\omega_X^1)_b^a = (\omega_X^M)_b^a = \Gamma_{cb}^a X^c$ .  
(ii)

$$\frac{1}{i|\xi|^{-2}\xi} \sigma_{-1}(\omega_X^1) = \frac{1}{2}(-2R(X, \dot{\gamma}) - R(\cdot, \dot{\gamma})X + R(X, \cdot)\dot{\gamma}).$$

Equivalently,

$$\frac{1}{i|\xi|^{-2}\xi} \sigma_{-1}(\omega_X^1)_b^a = \frac{1}{2}(-2R_{cdb}{}^a - R_{bdc}{}^a + R_{cbd}{}^a)X^c \dot{\gamma}^d.$$

*Proof.* (i) For  $\sigma_0(\omega_X^1)$ , the only term in (2.9) of order zero is the Christoffel term.

(ii) For  $\sigma_{-1}(\omega_X^1)$ , label the last six terms on the right hand side of (2.9) by (a), ..., (f). By Leibniz rule for the tensors, the only terms of order  $-1$  come from: in (a),  $-\nabla_{\dot{\gamma}}(R(X, \dot{\gamma})Y) = -R(X, \dot{\gamma})\nabla_{\dot{\gamma}}Y$  + lower order in  $Y$ ; in (b), the term  $-R(X, \dot{\gamma})\nabla_{\dot{\gamma}}Y$ ; in (c), the term  $-R(\nabla_{\dot{\gamma}}Y, \dot{\gamma})X$ ; in (e), the term  $R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma}$ .

For any vectors  $Z, W$ , the curvature endomorphism  $R(Z, W) : TM \rightarrow TM$  has

$$R(Z, W)_b^a = R_{cdb}^a Z^c W^d.$$

Also, since  $(\nabla_{\dot{\gamma}}Y)^a = \frac{d}{d\theta}Y^a$  plus zeroth order terms,  $\sigma_1(\nabla_{\dot{\gamma}}) = i\xi \cdot \text{Id}$ . Thus in (a) and (b),  $\sigma_1(-R(X, \dot{\gamma})\nabla_{\dot{\gamma}})_b^a = -R_{cdb}^a X^c \dot{\gamma}^d \xi$ .

For (c), we have  $-R(\nabla_{\dot{\gamma}}Y, \dot{\gamma})X = -R_{cdb}^a (\nabla_{\dot{\gamma}}Y)^c \dot{\gamma}^d X^b \partial_a$ , so the top order symbol is  $-R_{cdb}^a \xi \dot{\gamma}^d X^b = -R_{bdc}^a \xi \dot{\gamma}^d X^c$ .

For (e), we have  $R(X, \nabla_{\dot{\gamma}}Y)\dot{\gamma} = R_{cdb}^a X^c (\nabla_{\dot{\gamma}}Y)^d \dot{\gamma}^b \partial_a$ , so the top order symbol is  $R_{cdb}^a X^c \xi \dot{\gamma}^b = R_{cbd}^a X^c \xi \dot{\gamma}^d$ .

Since the top order symbol of  $(1 + \Delta)^{-1}$  is  $|\xi|^{-2}$ , adding these four terms finishes the proof.  $\square$

We now compute the top symbols of the curvature tensor.  $\sigma_{-1}(\Omega^1)$  involves the covariant derivative of the curvature tensor on  $M$ , but fortunately this symbol will not be needed in [10].

**Lemma A.2.** (i)  $\sigma_0(\Omega^1(X, Y))_b^a = R^M(X, Y)_b^a = R_{cdb}^a X^c Y^d$ .

(ii)

$$\begin{aligned} \frac{1}{i|\xi|^{-2}\xi} \sigma_{-1}(\Omega^1(X, Y)) &= \frac{1}{2} (\nabla_X [-2R(Y, \dot{\gamma}) - R(\cdot, \dot{\gamma})Y + R(Y, \cdot)\dot{\gamma}] \\ &\quad - (X \leftrightarrow Y) \\ &\quad - [-2R([X, Y], \dot{\gamma}) - R(\cdot, \dot{\gamma})[X, Y] + R([X, Y], \cdot)\dot{\gamma}]). \end{aligned}$$

Equivalently, in Riemannian normal coordinates on  $M$  centered at  $\gamma(\theta)$ ,

$$\begin{aligned} \frac{1}{i|\xi|^{-2}\xi} \sigma_{-1}(\Omega^1(X, Y))_b^a &= \frac{1}{2} X [(-2R_{cdb}^a - R_{bdc}^a + R_{cbd}^a) \dot{\gamma}^d] Y^c - (X \leftrightarrow Y) \\ &= \frac{1}{2} X [-2R_{cdb}^a - R_{bdc}^a + R_{cbd}^a] \dot{\gamma}^d Y^c - (X \leftrightarrow Y) \quad (\text{A.1}) \\ &\quad + \frac{1}{2} [-2R_{cdb}^a - R_{bdc}^a + R_{cbd}^a] \dot{X}^d Y^c - (X \leftrightarrow Y) \end{aligned}$$

*Proof.* (i)

$$\begin{aligned} \sigma_0(\Omega^1(X, Y))_b^a &= \sigma_0((d\omega^1 + \omega^1 \wedge \omega^1)(X, Y))_b^a \\ &= [(d\sigma_0(\omega^1) + \sigma_0(\omega^1) \wedge \sigma_0(\omega^1))(X, Y)]_b^a \\ &= [(d\omega^M + \omega^M \wedge \omega^M)(X, Y)]_b^a \\ &= R^M(X, Y)_b^a = R_{cdb}^a X^c Y^d. \end{aligned}$$

(ii) Since  $\sigma_0(\Omega_X^1)$  is independent of  $\xi$ , after dividing by  $i|\xi|^{-2}\xi$  we have

$$\begin{aligned} \sigma_{-1}(\Omega^1(X, Y))_b^a &= (d\sigma_{-1}(\omega^1)(X, Y))_b^a + \sigma_0(\omega_X^1)_c^a \sigma_{-1}(\omega_Y^1)_b^c + \sigma_{-1}(\omega_X^1)_c^a \sigma_0(\omega_Y^1)_b^c \\ &\quad - \sigma_0(\omega_Y^1)_c^a \sigma_{-1}(\omega_X^1)_b^c + \sigma_{-1}(\omega_Y^1)_c^a \sigma_0(\omega_X^1)_b^c. \end{aligned}$$

As an operator on sections of  $\gamma^*TM$ ,  $\Omega^{LM} - \Omega^M$  has order  $-1$  so  $\sigma_{-1}(\Omega^{LM}) = \sigma_{-1}(\Omega^{LM} - \Omega^M)$  is independent of coordinates. In Riemannian normal coordinates at  $\gamma(\theta)$ ,  $\sigma_0(\omega_X) = \sigma_0(\omega_Y) = 0$ , so

$$\begin{aligned} \sigma_{-1}(\Omega^1(X, Y))_b^a &= (d\sigma_{-1}(\omega^1)(X, Y))_b^a \\ &= X(\sigma_{-1}(\omega_Y^1))_b^a - Y(\sigma_{-1}(\omega_X^1))_b^a - \sigma_{-1}(\omega_{[X, Y]})_b^a \\ &= \frac{1}{2}X[(-2R_{cdb}^a - R_{bdc}^a + R_{cbd}^a)Y^c\dot{\gamma}^d] - (X \leftrightarrow Y) \\ &\quad - \frac{1}{2}(-2R_{cdb}^a - R_{bdc}^a + R_{cbd}^a)[X, Y]^c\dot{\gamma}^d. \end{aligned}$$

The terms involving  $X(Y^c) - Y(X^c) - [X, Y]^c$  cancel (as they must, since the symbol two-form cannot involve derivatives of  $X$  or  $Y$ ). Thus

$$\sigma_{-1}(\Omega^1(X, Y))_b^a = \frac{1}{2}X[(-2R_{cdb}^a - R_{bdc}^a + R_{cbd}^a)Y^c\dot{\gamma}^d] - (X \leftrightarrow Y).$$

This gives the first coordinate expression in (A.1). The second expression follows from  $X(\dot{\gamma}^d) = \dot{X}^d$  (see (2.8)).

To convert from the coordinate expression to the covariant expression, we follow the usual procedure of changing ordinary derivatives to covariant derivatives and adding bracket terms. For example,

$$\begin{aligned} \nabla_X(R(Y, \dot{\gamma})) &= (\nabla_X R)(Y, \dot{\gamma}) + R(\nabla_X Y, \dot{\gamma}) + R(Y, \nabla_X \dot{\gamma}) \\ &= X^i R_{cdb; i}^a Y^c \dot{\gamma}^d + R(\nabla_X Y, \dot{\gamma}) + R_{cdb}^a Y^c (\nabla_X \dot{\gamma})^d. \end{aligned}$$

In Riemannian normal coordinates at  $\gamma(\theta)$ , we have  $X^i R_{cdb; i}^a = X^i \partial_i R_{cdb}^a = X(R_{cdb}^a)$  and  $(\nabla_X \dot{\gamma})^d = X(\dot{\gamma}^d)$ . Thus

$$\nabla_X(R(Y, \dot{\gamma})) - (X \leftrightarrow Y) - R([X, Y], \dot{\gamma}) = X(R_{cdb}^a \dot{\gamma}^d)Y^c - (X \leftrightarrow Y).$$

The other terms are handled similarly.  $\square$

## A.2. Connection and Curvature Symbols for General $s$ .

The noteworthy feature of these computations is the linear dependence of  $\sigma_{-1}(\omega^s)$  on  $s$ .

Let  $g$  be the Riemannian metric on  $M$ .

**Lemma A.3.** (i) At  $\gamma(\theta)$ ,  $\sigma_0(\omega_X^s)_b^a = (\omega_X^M)_b^a = \Gamma_{cb}^a X^c$ .

(ii)  $\sigma_0(\Omega^s(X, Y))_b^a = R^M(X, Y)_b^a = R_{cdb}^a X^c Y^d$ .

(iii)  $\frac{1}{i|\xi|^{-2}\xi}\sigma_{-1}(\omega_X^s)_b^a = sT(X, \dot{\gamma}, g)$ , where  $T(X, \dot{\gamma}, g)$  is tensorial and independent of  $s$ .



*Proof.* (i) By Lemma 2.11, the only term of order zero in (2.26) is  $\omega_X^M$ .

(ii) The proof of Lemma A.2(ii) carries over.

(iii) By Theorem 2.12, we have to compute  $\sigma_{2s-1}$  for  $[D_X, (1+\Delta)^s]$ ,  $[D., (1+\Delta)^s]X$ , and  $[D., (1+\Delta)^s]X^*$ , as  $\sigma_{-1}((1+\Delta)^{-s}[D_X, (1+\Delta)^s]) = |\xi|^{-2s}\sigma_{-1}([D_X, (1+\Delta)^s])$ , etc.

Write  $D_X = \delta_X + \Gamma \cdot X$  in shorthand. Since  $(1+\Delta)^s$  has scalar leading order symbol,  $[\Gamma \cdot X, (1+\Delta)^s]$  has order  $2s-1$ . Thus we can compute  $\sigma_{2s-1}([\Gamma \cdot X, (1+\Delta)^s])$  in any coordinate system. In Riemannian normal coordinates centered at  $\gamma(\theta)$ , as in the proof of Lemma 2.11(ii), the Christoffel symbols vanish. Thus  $\sigma_{2s-1}([\Gamma \cdot X, (1+\Delta)^s]) = 0$ .

By (2.14),  $\sigma_{2s-1}([\delta_X, (1+\Delta)^s])$  is  $s$  times a tensorial expression in  $X, \dot{\gamma}, g$ , since  $\partial_i \Gamma_{\nu j}^\ell = \frac{1}{3}(R_{i\nu j}^\ell + R_{ij\nu}^\ell)$  in normal coordinates. The term with  $\Gamma$  vanishes, so  $\sigma_{2s-1}([D_X, (1+\Delta)^s])$  is  $s$  times this tensorial expression.

The argument for  $\sigma_{2s-1}([D., (1+\Delta)^s]X)$  is similar. The term with  $\Gamma$  vanishes. By (2.23), (2.24),

$$\sigma_{2s-1}([\delta., (1+\Delta)^s]X)_b^a = i \sum_j \partial_t^j \partial_\xi^j|_{t=0, \xi=0} (p(\theta, \xi)_e^a \Gamma_{bc}^e(\gamma - t, \eta + \xi) X^c(\theta - t)).$$

By (2.14), the right hand side is linear in  $s$  for  $\text{Re}(s) < 0$ . By (2.15), this implies the linearity in  $s$  for  $\text{Re}(s) > 0$ .

Since  $\sigma_{2s-1}([D., (1+\Delta)^s]X^*) = (\sigma_{2s-1}([D., (1+\Delta)^s]X))^*$ , this symbol is also linear in  $s$ .  $\square$

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